23. veljače 2022.

Popis pitanja:

OSNOVE FIZIČKIH MJERENJA I STATISTIČKE ANALIZE

III semestar akademske 2021. - 2022. godine

Na usmenom ispitu se izvlači po jedno pitanje iz svake skupine pitanja (A, B i C). Student(ica) može odmah odgovarati, a može najprije pisati koncept odgovora (ne dulje od 30 min.), a zatim odgovarati. Točan odgovor na svako pitanje nosi maksimalno po 100 bodova (dakle, sve skupa 300). Za prolaz na usmenom dijelu ispita, potrebno je ostvariti najmanje po 50 bodova za pitanje iz skupine A, 50 bodova za pitanje iz skupine B i 50 bodova za pitanje iz skupine C.

Da biste bili sigurni da gledate najsvježiju verziju ovog teksta, pritisnite na svojoj tipkovnici Shift + F5.

Za vrijeme ispita ne smijete koristiti mobitel.

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A skupina pitanja

1. Na temelju kombinatoričkih argumenata izvedite binomni teorem. Poopćite ga na izračun N-te potencije tri i četiri člana.

2. Definirajte pojmove „ili" i „i" vjerojatnosti. Navedite primjere. Opišite Bernoullijeve pokuse.

3. Izvedite Poincaréovu formulu. Navedite jedan primjer njezine primjene.

4. Izvedite Bayesov teorem i pomoću njega objasnite Mendelove pokuse križanja graška.

5. Navedite jedan primjer računa vjerojatnosti u problemima višestrukog izbora.

6. U kvantnoj mehanici je gustoća vjerojatnosti dana sa $\|\Psi\|^2$. Ako je elektron opisan valnom funkcijom $\Psi = \mathcal{N}\ e^{-r/a_0}$, odredite normiranje $\mathcal{N}$ i srednju udaljenost elektrona od jezgre.

7. Navedite primjer računa momenata u okviru statističke fizike i klasične mehanike.

8. Izračunajte marginalne i uvjetne vjerojatnosti dvodimenzijske raspodjele $\rho(x_1, x_2) = \mathcal{N}\ \exp\left[-\left(x_1^2 - 2\lambda x_1 x_2 + x_2^2\right)/(2\sigma^2)\right]$.

9. U kontekstu preobrazbe kontinuirane nasumične varijable, povežite raspodjelu čestica klasičnog idealnog plina po brzinama, s raspodjelom istih čestica po energijama.

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B skupina pitanja

1. Definirajte Geometrijsku raspodjelu; skicirajte ju; pokažite normiranje; izračunajte prva dva momenta; izračunajte kumulativnu funkciju.
2. Definirajte Binomnu raspodjelu; skicirajte ju; pokažite normiranje; izračunajte prva dva momenta; izračunajte kumulativnu funkciju.
3. Definirajte Poissonovu raspodjelu; skicirajte ju; pokažite normiranje; izračunajte prva dva momenta; izračunajte kumulativnu funkciju.
4. Polazeći od binomne raspodjele, izvedite Gaußovu raspodjelu; skicirajte ju; pokažite normiranje; izračunajte prva dva momenta; izračunajte kumulativnu funkciju; pokažite adicijski teorem.
5. Definirajte gama raspodjelu; skicirajte ju; pokažite normiranje; izračunajte prva dva momenta; izračunajte kumulativnu funkciju; pokažite vezu s Maxwell-Boltzmannovom raspodjelom.
6. Polazeći od načela maksimuma entropije, izvedite geometrijsku raspodjelu.
7. Polazeći od načela maksimuma entropije, izvedite Gaußovu raspodjelu.
8. Izvedite funkciju izvodnicu momenata binomne raspodjele. Izračunajte varijancu.
9. Izvedite funkciju izvodnicu momenata Poissonove raspodjele. Izračunajte varijancu.
10. Izvedite funkciju izvodnicu momenata Gaußove raspodjele. Pomoću nje izračunajte treći i četvrti moment.
11. Izvedite funkciju izvodnicu momenata gama raspodjele. Izračunajte varijancu.
12. Izvedite teorem inverzije. Polazeći od karakteristične funkcije Cauchyjeve raspodjele, izračunajte pridruženu gustoću vjerojatnosti.
13. Izvedite karakterističnu funkciju binomne raspodjele. Izračunajte varijancu.
14. Izvedite karakterističnu funkciju Poissonove raspodjele. Izračunajte varijancu.
15. Izvedite karakterističnu funkciju Gaußove raspodjele. Pomoću nje izračunajte treći i četvrti moment.
16. Izvedite karakterističnu funkciju gama raspodjele. Izračunajte varijancu.
17. Iskažite i dokažite središnji granični teorem.

C skupina pitanja

1. Dokažite da je aritmetička sredina niza mjerenja najvjerojatnije najbliža točnoj vrijednosti mjerene veličine.

2. Ako su poznate procjene grešaka varijabla $x, y, z, \cdots$, procjenite grešku funkcije $f(x, y, z, \cdots)$. Primjenite ovaj rezultat na procjenu pogreške aritmetičke sredine.

3. Za niz podatka u obliku parova $[x(n), y(n)]$, izračunajte koeficijente pravca $y = a_0 + a_1 x$ koji, u smislu najmanjeg kvadratnog odstupanja, prolazi najbliže zadanim točkama.

4. Nizom mjerenja su dobiveni parovi točaka $[x(n), y(n)]$. Izračunajte koeficijente kvadratne funkcije $y = a_0 + a_1 x + a_2 x^2$ koja, u smislu najmanjeg kvadratnog odstupanja, prolazi najbliže izmjerenim točkama.

5. Definirajte sva tri korelacijska pravca. Za jedan od njih, izvedite izraze za parametre pravca.

6. Izvedite izraz za koeficijent korelacije i objasnite njegovo značenje kao mjere korelacije.

7. Definirajte korelacijski polinom i izvedite izraze za parametre polinoma.

8. Izvedite izraz za indeks korelacije i objasnite njegovo značenje kao mjere korelacije.

9. Objasnite smisao i razloge primjene uzoraka u statističkoj analizi. Pokažite da je aritmetička sredina od aritmetičke sredine svih uzoraka jednaka aritmetičkoj sredini cijelog osnovnog skupa.

10. Objasnite testiranje hipoteze pomoću χ^2 testa.

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Matematički dodatak

1 Trigonometrija

$$\sinh^2\left(\frac{x}{2}\right) = \frac{\cosh(x) - 1}{2} \qquad \cosh^2\left(\frac{x}{2}\right) = \frac{\cosh(x) + 1}{2}.$$

$$\begin{aligned} \cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) &= -\frac{1}{2}, \\ \cos^2\left(\frac{2\pi}{5}\right) + \cos^2\left(\frac{4\pi}{5}\right) &= \frac{3}{4}, \\ \cos\left(\frac{2\pi}{5}\right) \cdot \cos\left(\frac{4\pi}{5}\right) &= -\frac{1}{4}. \end{aligned}$$

$$\cosh(\alpha + \beta) \cdot \cosh(\alpha - \beta) = \frac{1}{2}\left[\cosh(2\alpha) + \cosh(2\beta)\right],$$

$$\tanh(x \pm y) = \frac{\tanh(x) \pm \tanh y}{1 \pm \tanh(x)\,\tanh y},$$

$$\sin(\alpha) = \cos(\pi/2 - \alpha).$$

2 Redovi

$$f(x, y, z) = f(x_0, y_0, z_0) + \sum_{n=1}^{\infty} \frac{1}{n!}\left[(x - x_0)\frac{\partial}{\partial x} + (y - y_0)\frac{\partial}{\partial y} + (z - z_0)\frac{\partial}{\partial z}\right]^n f(x, y, z)\bigg|_{x_0, y_0, z_0}. \tag{1}$$

Iz razvoja eksponencijalne funkcije

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \tag{2}$$

posredstvom Eulerove formule

$$e^{i\alpha} = \cos(\alpha) + i\,\sin(\alpha), \qquad i^2 = -1, \tag{3}$$

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dolazi se do razvoja trigonometrijskih i hiperbolnih funkcija

$$\sin(x) = x - \frac{x^3}{6} + \cdots = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)\,!} \tag{4}$$

$$\cos(x) = 1 - \frac{x^2}{2} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)\,!} \tag{5}$$

$$\sinh(x) = x + \frac{x^3}{6} + \frac{x^5}{120} + \mathcal{O}\left(x^7\right), \qquad -\infty < x < +\infty \tag{6}$$

$$\cosh(x) = 1 + \frac{x^2}{2} + \frac{x^4}{24} + \mathcal{O}\left(x^6\right), \qquad -\infty < x < +\infty \tag{7}$$

$$\tanh(x) = x - \frac{x^3}{3} + \frac{2\,x^5}{15} + \mathcal{O}\left(x^7\right), \tag{8}$$

$$(1 \pm \alpha)^N = 1 \pm N\,\alpha + \binom{N}{2}\alpha^2 \pm \binom{N}{3}\alpha^3 + \cdots \tag{9}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n+1}\frac{x^n}{n} + \cdots \qquad -1 < x \le 1, \tag{10}$$

$$\ln(1-x) = -\left[x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n} + \cdots\right] \qquad -1 \le x < 1, \tag{11}$$

$$\ln\left(\frac{1+x}{1-x}\right) = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2n+1}}{2n+1} + \cdots\right] \qquad -1 < x < 1, \tag{12}$$

$$\ln\left[\cosh(x)\right] = \frac{1}{2}\,x^2 - \frac{1}{12}\,x^4 + \frac{1}{45}\,x^6 + \cdots \qquad x \to 0, \tag{13}$$

$$\frac{1}{\sqrt{1 \pm x}} = 1 \mp \frac{1}{2}\,x + \frac{1\cdot 3}{2\cdot 4}\,x^2 \mp \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\,x^3 + \cdots \qquad |x| < 1 \tag{14}$$

$$\frac{1}{1 \pm x} = 1 \mp x + x^2 \mp x^3 + x^4 \mp \cdots \qquad |x| < 1 \tag{15}$$

$$\frac{1}{(1 \pm x)^{3/2}} = 1 \mp \frac{3}{2}\,x + \frac{3\cdot 5}{2\cdot 4}\,x^2 \mp \frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6}\,x^3 \cdots \qquad |x| < 1 \tag{16}$$

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$$\sum_{n=1}^{N} q^{n-1} = \frac{q^N - 1}{q - 1}, \tag{17}$$

$$\sum_{n=0}^{N} q^{n} = \frac{q^{N+1} - 1}{q - 1}, \tag{18}$$

$$\sum_{n=0}^{N-1} (a + nr)\, q^{n} = \frac{a - [a + (N-1)r]q^N}{1 - q} + qr\frac{1 - q^{N-1}}{(1-q)^2}, \tag{19}$$

$$\sum_{n=1}^{N} [a_1 + (n-1)d] = \frac{1}{2} N(a_1 + a_N) \tag{20}$$

$$\sum_{n=0}^{\infty} q^{n} = \frac{1}{1 - q} \qquad |q| < 1, \tag{21}$$

$$\sum_{n=0}^{\infty} n \cdot q^{n} = \sum_{n=1}^{\infty} n \cdot q^{n} = \frac{q}{(1-q)^2} \qquad |q| < 1, \tag{22}$$

$$\sum_{m=1}^{M} \cos\left(\frac{2\pi \cdot m \cdot l}{N}\right) = -\frac{1}{2}, \quad N = 2M + 1, \quad l = 1, 2, \cdots, M$$

$$\sum_{m=1}^{M} \cos\left(\frac{2\pi \cdot m \cdot k}{N}\right) \cos\left(\frac{2\pi \cdot m \cdot l}{N}\right) = \begin{cases} \frac{N-2}{4} & k = l, \\ -\frac{1}{2} & k \neq l\,. \end{cases}$$

$$N = 2M + 1, \quad k = 1, 2, \cdots, M, \quad l = 1, 2, \cdots, M$$

$$\sum_{m=1}^{M} \sin\left(\frac{2\pi \cdot m \cdot k}{N}\right) \sin\left(\frac{2\pi \cdot m \cdot l}{N}\right) = \begin{cases} -\frac{N}{4} & k + l = N, \\ 0 & k + l \neq N\,. \end{cases}$$

$$N = 2M + 1, \quad l = 1, 2, \cdots, M, \quad k = M + 1, M + 2, \cdots, 2M.$$

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3 Neodređeni integrali

$$\int \left(1 - x^{2/3}\right)^{3/2} \mathrm{d}x = \frac{1}{16}\left[\sqrt{1-x^{2/3}}\left(-8x^{5/3} + 14x - 3x^{1/3}\right) + 3\arcsin\left(x^{1/3}\right)\right] \tag{23}$$

$$\int \sqrt{x^2 + a^2}\, \mathrm{d}x = \frac{1}{2}\left[x\sqrt{x^2+a^2} + a^2 \ln\left(x + \sqrt{x^2+a^2}\right)\right] \tag{24}$$

$$\int \frac{x\, \mathrm{d}x}{ax+b} = \frac{x}{a} - \frac{b}{a^2}\ln|ax+b|, \tag{25}$$

$$\int \frac{\mathrm{d}x}{a+bx+cx^2} = \frac{2}{\sqrt{4ac-b^2}} \arctan\left(\frac{b+2cx}{\sqrt{4ac-b^2}}\right), \tag{26}$$

$$\int \frac{\mathrm{d}x}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| = \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right), \qquad |x| < a, \tag{27}$$

$$\int \frac{\mathrm{d}x}{a^2+x^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right), \tag{28}$$

$$\int \frac{x\, \mathrm{d}x}{a^2 \pm x^2} = \pm\frac{1}{2}\ln\left(a^2 \pm x^2\right), \tag{29}$$

$$\int \frac{x\, \mathrm{d}x}{a^3 \pm x^3} = \frac{1}{6a}\ln\frac{a^2 \mp ax + x^2}{(a \pm x)^2} \pm \frac{1}{a\sqrt{3}}\arctan\frac{2x \mp a}{a\sqrt{3}}, \tag{30}$$

$$\int \frac{\mathrm{d}x}{a^4-x^4} = \frac{1}{4a^3}\ln\left|\frac{a+x}{a-x}\right| + \frac{1}{2a^3}\arctan\left(\frac{x}{a}\right), \tag{31}$$

$$\int \frac{\mathrm{d}x}{\sqrt{a^2+x^2}} = \ln\left(x + \sqrt{a^2+x^2}\right) = \mathrm{Ar}\sinh\left(\frac{x}{a}\right) \tag{32}$$

$$\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} = \arcsin\left(\frac{x}{a}\right), \tag{33}$$

$$\int \frac{x^2\, \mathrm{d}x}{\sqrt{a^2-x^2}} = -\frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right), \tag{34}$$

$$\int \frac{\mathrm{d}x}{\sqrt{a^3 \pm x^3}} = \pm\frac{1}{6a^2}\ln\left[\frac{(a \pm x)^2}{a^2 \mp ax + x^2}\right] + \frac{1}{a^2\sqrt{3}}\,\mathrm{atan}\left(\frac{2x \mp a}{a\sqrt{3}}\right), \tag{35}$$

$$\int \frac{\mathrm{d}x}{(a^2+x^2)^{3/2}} = \frac{x}{a^2\left(x^2+a^2\right)^{1/2}}, \tag{36}$$

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$$\int \frac{dx}{\sqrt{\alpha x^2+\beta x+\gamma}} = \begin{cases} \frac{1}{\sqrt{\alpha}}\dots, & \alpha > 0 \\ \frac{1}{\sqrt{\alpha}}\dots, & \alpha > 0, \Delta > 0 \\ \frac{1}{\sqrt{\alpha}}\dots, & \alpha > 0, \Delta = 0 \\ \frac{-1}{\sqrt{-\alpha}} \arcsin\left(\frac{2\alpha x+\beta}{\sqrt{-\Delta}}\right), & \alpha < 0, \Delta < 0 \end{cases} \tag{37}$$

$$\int \frac{x\,dx}{\sqrt{a\,x^2+b\,x+c}} = \frac{\sqrt{a\,x^2+b\,x+c}}{a} - \frac{b}{2a}\int \frac{dx}{\sqrt{a\,x^2+b\,x+c}} \tag{38}$$

$$\int \frac{dx}{x\sqrt{\alpha x^2+\beta x+\gamma}} = \frac{1}{\sqrt{-\gamma}} \arcsin\left(\frac{\beta x+2\gamma}{x\sqrt{-\Delta}}\right), \quad \Delta = 4\alpha\gamma-\beta^2, \tag{39}$$

$$\int \cos^4(x)\,dx = \frac{3}{8}x + \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) \tag{40}$$

$$\int e^{ax}\sin(bx)\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin(bx) - b\cos(bx)\right]. \tag{41}$$

$$\int \frac{dx}{p+q\,\sin(ax)} = \begin{cases} \frac{2}{a\sqrt{p^2-q^2}} \arctan\left[\frac{p\tan\left(\frac{ax}{2}\right)+q}{\sqrt{p^2-q^2}}\right], & p^2 > q^2 \\ \frac{1}{a\sqrt{q^2-p^2}} \ln\left[\frac{p\tan\left(\frac{ax}{2}\right)+q-\sqrt{q^2-p^2}}{p\tan\left(\frac{ax}{2}\right)+q+\sqrt{q^2-p^2}}\right], & p^2 < q^2. \end{cases} \tag{42}$$

$$\int \frac{dx}{b+c\,\cos(ax)} = \begin{cases} \frac{2}{a\sqrt{b^2-c^2}} \arctan\left[\frac{(b-c)\tan\left(\frac{ax}{2}\right)}{\sqrt{b^2-c^2}}\right], & b^2 > c^2 \\ \frac{1}{a\sqrt{c^2-b^2}} \ln\left[\frac{(c-b)\tan\left(\frac{ax}{2}\right)+\sqrt{c^2-b^2}}{(c-b)\tan\left(\frac{ax}{2}\right)-\sqrt{c^2-b^2}}\right], & b^2 < c^2. \end{cases} \tag{43}$$

$$\int x^n e^{ax}\,dx = \frac{x^n}{a}e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx \tag{44}$$

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4 Određeni integrali

$$\int_0^\infty x^n \, e^{-ax} \, dx = \frac{\Gamma(n+1)}{a^{n+1}}. \tag{45}$$

$$\int_{-\infty}^{+\infty} e^{-ax^2+bx} \, dx = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a}\right), \qquad a > 0, \tag{46}$$

$$\int_{-\infty}^{+\infty} x \, e^{-ax^2+bx} \, dx = \frac{b\sqrt{\pi}}{2\,a^{3/2}} \exp\left(\frac{b^2}{4a}\right), \qquad \mathrm{Re}(a) > 0, \tag{47}$$

$$\int_{-\infty}^{+\infty} x^2 \, e^{-ax^2+bx} \, dx = \frac{(b^2+2a)\sqrt{\pi}}{4\,a^{5/2}} \exp\left(\frac{b^2}{4a}\right), \qquad \mathrm{Re}(a) > 0, \tag{48}$$

$$\int_{-\infty}^{+\infty} x^3 \, e^{-ax^2+bx} \, dx = \frac{b(b^2+6a)\sqrt{\pi}}{8\,a^{7/2}} \exp\left(\frac{b^2}{4a}\right), \qquad \mathrm{Re}(a) > 0, \tag{49}$$

$$\int_0^{+\infty} x^n \, e^{-ax^2} \, dx = \frac{\Gamma\left(\frac{n+1}{2}\right)}{2\,a^{(n+1)/2}}, \qquad a > 0, \quad n > -1$$

$$= \begin{cases} \dfrac{1\cdot 3\cdot \ldots \cdot (2k-1)}{2^{k+1}a^{k+1/2}} \sqrt{\pi} & n = 2k \\[2ex] \dfrac{k\,!}{2a^{k+1}} & n = 2k+1. \end{cases} \tag{50}$$

$$\int_{-\infty}^{+\infty} x^n \, e^{-ax^2+bx} \, dx = \sqrt{\frac{\pi}{a}} \, e^{b^2/(4a)} \sum_{k=0}^{[n/2]} \binom{n}{2k} (2k-1)!! \, (2a)^{k-n} \, b^{n-2k}, \quad a > 0, \tag{51}$$

$$\int_{-\infty}^{+\infty} x^k \, e^{-a\,x^4} \, dx = \frac{1}{2} \frac{\Gamma\left(\frac{k+1}{4}\right)}{a^{(k+1)/4}}, \qquad a > 0 \tag{52}$$

$$\int_0^{+\infty} e^{-ax^b} \, dx = \frac{\Gamma(1/b)}{b\,a^{1/b}}, \tag{53}$$

$$\int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2}\sum_{i,j=1}^{n} x_i A_{i,j} x_j + \sum_{i=1}^{n} B_i x_i\right) d^n x = \sqrt{\frac{(2\pi)^n}{\mathrm{Det}\,\mathbb{A}}} \exp\left(\frac{1}{2}\vec{B}^T\,\mathbb{A}^{-1}\,\vec{B}\right), \tag{54}$$

$$\int_0^{+\infty} x^{a-1}\sin(x)\ \mathrm{d}x = \sin\left(\frac{\pi a}{2}\right)\ \Gamma(a), \qquad -1 < Re(a) < 1 \tag{55}$$

$$\int_0^{+\infty} x^{a-1}\cos(x)\ \mathrm{d}x = \cos\left(\frac{\pi a}{2}\right)\ \Gamma(a), \qquad 0 < Re(a) < 1 \tag{56}$$

$$\int_0^{+\infty} \mathrm{e}^{-\lambda x^2}\cos(kx)\ \mathrm{d}x = \frac{1}{2}\sqrt{\frac{\pi}{\lambda}}\ \mathrm{e}^{-k^2/(4\lambda)}, \qquad \lambda > 0 \tag{57}$$

$$\int_0^{+\infty} \mathrm{e}^{-\lambda x^2}\frac{\sin(kx)}{x}\ \mathrm{d}x = \frac{\pi}{2}\ erf\left(\frac{k}{2\sqrt{\lambda}}\right) \tag{58}$$

$$\int_0^{+\infty} \frac{x^{a-1}}{1+x^b}\ \mathrm{d}x = \frac{\pi}{b\ \sin(\frac{a\pi}{b})}, \qquad 0 < a < b, \tag{59}$$

$$\int_0^{\infty} \frac{x^{\mu-1}\ \mathrm{d}x}{(p+qx^{\nu})^{n+1}} = \frac{1}{\nu\, p^{n+1}}\left(\frac{p}{q}\right)^{\mu/\nu}\frac{\Gamma(\mu/\nu)\,\Gamma(n+1-\mu/\nu)}{\Gamma(n+1)}, \quad 0 < \mu/\nu < n+1 \tag{60}$$

$$\int_0^{\pi/2} [\sin(x)]^{2n}\ \mathrm{d}x = \frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\cdots(2n)}\ \frac{\pi}{2} = \frac{(2n-1)\,!!}{(2n)\,!!}\ \frac{\pi}{2}, \tag{61}$$

$$\int_0^{\pi} [\sin(x)]^{2n+1}\ \mathrm{d}x = 2\frac{2\cdot4\cdot6\cdots(2n)}{3\cdot5\cdots(2n-1)(2n+1)} = 2\frac{(2n)\,!!}{(2n+1)\,!!}, \tag{62}$$

$$\int_0^{+\infty} \frac{\sin^2(ax)}{x^2}\ \mathrm{d}x = \frac{\pi}{2}\ |a|. \tag{63}$$

Integrali trigonometrijskih funkcija - najčešće zamjene

$$\int \sin^m(x)\ \cos^n(x)\ \mathrm{d}x \tag{64}$$

$$(1)\cos(x) = t, \qquad m > 0,\ \text{neparan}, \tag{65}$$

$$(2)\sin(x) = t, \qquad n > 0,\ \text{neparan}, \tag{66}$$

$$(3)\tan x = t, \qquad m+n < 0,\ \text{parno}. \tag{67}$$

Univerzalna trigonometrijska zamjena

$$\sin(x) = \frac{2t}{1+t^2}, \qquad \cos(x) = \frac{1-t^2}{1+t^2}, \qquad \mathrm{d}x = \frac{2\ \mathrm{d}t}{1+t^2}.$$

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5 Vektorski diferencijalni operatori

$$
\begin{aligned}
\vec{\nabla} s &= \left(\vec{e}_\rho \, \frac{\partial}{\partial \rho} + \frac{\vec{e}_\varphi}{\rho} \, \frac{\partial}{\partial \varphi} + \vec{e}_z \, \frac{\partial}{\partial z} \right) \, s, \\
\vec{\nabla} \vec{V} &= \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \, V_\rho \right) + \frac{1}{\rho} \frac{\partial \, V_\varphi}{\partial \varphi} + \frac{\partial \, V_z}{\partial z}, \\
\vec{\nabla} \times \vec{V} &= \frac{\vec{e}_\rho}{\rho} \, \left(\frac{\partial \, V_z}{\partial \varphi} - \rho \frac{\partial \, V_\varphi}{\partial z} \right) + \vec{e}_\varphi \, \left(\frac{\partial \, V_\rho}{\partial z} - \frac{\partial \, V_z}{\partial \rho} \right) + \frac{\vec{e}_z}{\rho} \, \left(\frac{\partial \, V_\varphi \rho}{\partial \rho} - \frac{\partial \, V_\rho}{\partial \varphi} \right), \\
\vec{\nabla}^2 s &= \left[\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \, \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2} \right] \, s
\end{aligned}
$$

$$
\begin{aligned}
\vec{\nabla} \, s &= \left(\vec{e}_r \, \frac{\partial}{\partial r} + \frac{\vec{e}_\theta}{r} \, \frac{\partial}{\partial \theta} + \frac{\vec{e}_\varphi}{r \sin(\theta)} \, \frac{\partial}{\partial \varphi} \right) \, s, \\
\vec{\nabla} \vec{V} &= \frac{1}{r^2} \, \frac{\partial}{\partial r} \left(r^2 \, V_r \right) + \frac{1}{r \sin(\theta)} \, \frac{\partial}{\partial \theta} \left[\sin(\theta) \, V_\theta \right] + \frac{1}{r \sin(\theta)} \, \frac{\partial \, V_\varphi}{\partial \varphi}, \\
\vec{\nabla} \times \vec{V} &= \frac{\vec{e}_r}{r \sin(\theta)} \, \left[\frac{\partial \, V_\varphi \sin(\theta)}{\partial \theta} - \frac{\partial \, V_\theta}{\partial \varphi} \right] + \frac{\vec{e}_\theta}{r \sin(\theta)} \, \left[\frac{\partial \, V_r}{\partial \varphi} - \sin(\theta) \frac{\partial \, r \, V_\varphi}{\partial r} \right] + \frac{\vec{e}_\varphi}{r} \, \left[\frac{\partial \, r \, V_\theta}{\partial r} - \frac{\partial \, V_r}{\partial \theta} \right], \\
\vec{\nabla}^2 s &= \left\{ \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin(\theta)} \frac{\partial}{\partial \theta} \left[\sin(\theta) \frac{\partial}{\partial \theta} \right] + \frac{1}{r^2 \sin^2(\theta)} \frac{\partial^2}{\partial \varphi^2} \right\} \, s.
\end{aligned}
$$

6 Gaußian or Stratonovich-Hubbard transformation

$$
\mathrm{e}^{a^2} = \sqrt{\frac{N}{2\,\pi}} \, \int_{-\infty}^{+\infty} \, \mathrm{e}^{-x^2 \, N/2 + a \, x \, \sqrt{2\,N}} \, \mathrm{d}x \, . \tag{68}
$$

http://en.wikipedia.org/wiki/Hubbard-Stratonovich_transformation

- Canning, JPA, **25**, 1992, 4723-35, Canning, JPA, **26**, 1993, 3029-36.
- A. Campa, T. Dauxois, S. Ruffo *Statistical mechanics and dynamics of solvable models with long-range interactions* Phys. Rev. **480** (2009) 57-159
- A. Campa, A Giansanti, d Moroni *Canonical solution of classical magnetic models with long-range couplings* J. Phys. A **36** (2003) 6897 - Nishimori, Ch 2

$$
\mathrm{e}^{c\,y^2/2} = \frac{1}{\sqrt{2\,\pi\,c}} \, \int_{-\infty}^{+\infty} \, \mathrm{e}^{-x^2/(2\,c) + xy} \, \mathrm{d}x \, , \tag{69}
$$

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$$e^{b^2/(4a)} = \sqrt{\frac{a}{\pi}} \int_{-\infty}^{+\infty} e^{-a\phi^2+b\,\phi}\, d\phi\,, \tag{70}$$

$$e^{a^2} = \frac{1}{\sqrt{2\,\pi}} \int_{-\infty}^{+\infty} e^{-\phi^2/2+a\,\phi\,\sqrt{2}}\, d\phi\,, \tag{71}$$

$$e^{a^2} = \sqrt{\frac{N}{2\,\pi}} \int_{-\infty}^{+\infty} e^{-x^2\,N/2+a\,x\,\sqrt{2\,N}}\, dx \tag{72}$$

$$e^{-a\,x^2/2} = \sqrt{\frac{1}{2\,\pi\,a}} \int_{-\infty}^{+\infty} e^{-y^2/(2\,a)-\imath\,y\,x}\, dy\,, \tag{73}$$

$$e^{a\,S^2} = \sqrt{\frac{1}{4\,\pi\,a}} \int_{-\infty}^{+\infty} e^{-z^2/(4\,a)+S\,z}\, dz\,, \qquad Re\ a > 0, \tag{74}$$

$$e^{b\,m^2} = \sqrt{\frac{b}{\pi}} \int_{-\infty}^{+\infty} e^{-b\,x^2+2\,m\,b\,x}\, dx\,, \tag{75}$$

$$e^{a\,x^2/2} = \sqrt{\frac{aN}{2\pi}} \int_{-\infty}^{+\infty} e^{-aNm^2/2+amx\sqrt{N}}\, dm\,, \tag{76}$$

$$e^{\frac{1}{2}\,Ks^2} = \sqrt{\frac{K}{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\,K\phi^2+Ks\phi}\, d\phi\,, \tag{77}$$

$$e^{\frac{1}{2}\,\sum_{i,j}\,K_{i,j}s_is_j} = \sqrt{\frac{\mathrm{Det}\,\mathbf{K}}{(2\pi)^N}} \int_{-\infty}^{+\infty} d\phi_1\cdots d\phi_N\ e^{-\frac{1}{2}\,\sum_{i,j}\,\phi_iK_{i,j}\phi_j+\sum_{i,j}\,s_iK_{i,j}\phi_j}. \tag{78}$$

$$e^{\frac{1}{2}\,\sum_{i,j}\,K_{i,j}s_is_j} = \sqrt{\frac{1}{\mathrm{Det}\,\mathbf{K}(2\pi)^N}} \int_{-\infty}^{+\infty} d\phi_1\cdots d\phi_N\ e^{-\frac{1}{2}\,\sum_{i,j}\,\phi_iK^{-1}_{i,j}\phi_j+\sum_{i,j}\,s_i\phi_j}. \tag{79}$$

♠

$$S_i\,S_j \;\to\; \left(\frac{1}{N}\sum_{i=1}^{N} S_i\right)^2,$$

7 Koordinatni sustavi

Cilindrični koordinatni sustavi

(a) kružni cilindar

$$x = \rho\ \cos(\varphi), \qquad y = \rho\ \sin(\varphi).$$

(b) eliptički cilindar

$$x = a\ \cosh(u)\ \cos(v), \qquad y = a\ \sinh(u)\ \sin(v).$$

(c) parabolički cilindar

$$x = \xi\,\eta, \qquad y = \frac{1}{2}\left(\eta^2 - \xi^2\right).$$

(d) bipolarni

$$x = \frac{a\ \sinh(\eta)}{\cosh(\eta) - \cos(\xi)}, \qquad y = \frac{a\ \sin(\eta)}{\cosh(\eta) - \cos(\xi)}.$$

POOPĆENI SFERNI KOORDINATNI SUSTAV

Sferni koordinatni sustav se može i poopćiti s tri dimenzije na proizvoljan broj dimenzija d. Neka su, umjesto s x, y, z, pravokutne koordinate označene s

$$x_1, x_2, \cdots, x_d,$$

a

$$R, \theta_1, \theta_2, \cdots, \theta_{d-1}$$

neka su sferne koordinate u d-dimenzijskom prostoru. Veza među njima je dana relacijama

$$\begin{aligned}
x_1 &= R\ \cos(\theta_1), \\
x_2 &= R\ \sin(\theta_1)\ \cos(\theta_2), \\
x_3 &= R\ \sin(\theta_1)\ \sin(\theta_2)\ \cos(\theta_3), \\
&\vdots \\
x_{d-1} &= R\ \sin(\theta_1)\ \sin(\theta_2)\ \cdots\ \sin(\theta_{d-2}) cos(\theta_{d-1}), \\
x_d &= R\ \sin(\theta_1)\ \sin(\theta_2)\ \cdots\ \sin(\theta_{d-2}) \sin(\theta_{d-1}).
\end{aligned}$$

Odgovarajući integrali se računaju kao

$$\int \mathrm{d}^d x = \int \mathrm{d}\Omega_d \int_0^\infty R^{d-1}\ \mathrm{d}R = \int \mathrm{d}\Omega_{d-1}\ [\sin(\theta_{d-1})]^{d-2}\ \mathrm{d}\theta_{d-1} \int_0^\infty R^{d-1}\ \mathrm{d}R,$$

$$\begin{aligned}
\int \mathrm{d}\Omega_d &= \int_0^{2\pi} \mathrm{d}\theta_1 \int_0^\pi \sin(\theta_2)\ \mathrm{d}\theta_2 \cdots \int_0^\pi [\sin(\theta_{d-1})]^{d-2}\ \mathrm{d}\theta_{d-1} = \frac{2\,\pi^{d/2}}{\Gamma(d/2)} \\
&= \Omega_{d-1} \int_0^\pi [\sin(\theta_{d-1})]^{d-2}\ \mathrm{d}\theta_{d-1}
\end{aligned}$$

$$\begin{aligned}
\Omega_1 &= 2, \\
\Omega_2 &= 2\,\pi, \\
\Omega_3 &= 4\,\pi, \\
\Omega_4 &= 2\,\pi^2, \\
\Omega_5 &= \frac{8}{3}\,\pi^2, \\
\Omega_6 &= \pi^3, \\
\Omega_7 &= \frac{16}{15}\,\pi^3, \\
\Omega_8 &= \frac{1}{3}\,\pi^4, \\
\Omega_9 &= \frac{32}{105}\,\pi^4, \\
\Omega_{10} &= \frac{1}{12}\,\pi^5.
\end{aligned}$$

8 Neke specijalne funkcije

RIEMANNOVA ζ FUNKCIJA

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \left(1 - 2^{1-s} \right) \zeta(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \cdots$$

Lerch

$$\zeta(a,s) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, \qquad \zeta(1,s) = \zeta(s),$$

$$\frac{1}{p^s} \sum_{n=1}^{p} \zeta\left(\frac{n}{p}, s\right) = \zeta(s).$$

BESSELOVA FUNKCIJA NULTOG REDA

$$J_0(z) = \int_0^{2\pi} \mathrm{d}\theta \; \mathrm{e}^{\imath\, z_1 \cos\theta + \imath\, z_2 \sin\theta} = \int_0^{2\pi} \mathrm{d}\theta \; \mathrm{e}^{\imath\, z \cos\theta}, \qquad z = \sqrt{z_1^2 + z_2^2}.$$

Modificirana Besselova funkcija reda 0

$$I_0(z) = \int_0^{2\pi} \mathrm{d}\theta \; \mathrm{e}^{z_1 \cos\theta + z_2 \sin\theta} = \int_0^{2\pi} \mathrm{d}\theta \; \mathrm{e}^{z \cos\theta}, \qquad z = \sqrt{z_1^2 + z_2^2}.$$

GAMA FUNKCIJA

$$\Gamma(n) = (n-1)!$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \;\; \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}, \;\; \Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi},$$

APROKSIMATIVNA STIRLINGOVA FORMULA

$$\ln\,(n\,!) = n\,\ln(n) - n + \frac{1}{2}\,\ln\,(2\,\pi\,n) + \frac{1}{12\,n} - \frac{1}{360\,n^3} + \cdots. \tag{80}$$

DIRACOVA δ-FUNKCIJA

$$\delta(x-a) = 0, \qquad x \neq a,$$

$$\int f(x)\,\delta(x-a)\,\,\mathrm{d}x = f(a),$$

$$\int_{-\infty}^{\infty} f(x)\,\delta[g(x)]\,\,\mathrm{d}x = \sum_{n=1}^{N} \frac{1}{|g'(x_n)|}\,f(x_n)$$

$$\delta[g(x)] = \sum_{n=1}^{N} \frac{\delta(x-x_n)}{|g'(x_n)|} \qquad\qquad g(x_n) = 0,\; g'(x_n) \neq 0$$

$$\delta(t-x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{\imath\,\omega(t-x)}\,\,\mathrm{d}\omega\,.$$

9 Razno

♣ Jakobijan prijelaza iz jednog u drugi koordinatni sustav

$$\mathrm{d}x_1\;\mathrm{d}x_2\;\cdots\;\mathrm{d}x_N = \mathrm{Det}\begin{bmatrix} \frac{\partial x_1}{\partial x_1'} & \cdots & \frac{\partial x_1}{\partial x_N'} \\ \vdots & \vdots & \vdots \\ \frac{\partial x_N}{\partial x_1'} & \cdots & \frac{\partial x_N}{\partial x_N'} \end{bmatrix}\;\mathrm{d}x_1'\;\mathrm{d}x_2'\;\cdots\;\mathrm{d}x_N'\,.$$

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♣ Hesse-ova matrica

$$H_{ij}[f(x)] = \mathrm{d}_i \mathrm{d}_j f(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2}. \end{bmatrix}$$

RANG matrice je broj njezinih svojstvenih vrijednosti različitih od nule.

♣ Leibnitz: formula za derivaciju određenog integrala

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} \int_{g(\alpha)}^{h(\alpha)} f(x,\alpha)\ \mathrm{d}x = \int_{g(\alpha)}^{h(\alpha)} \frac{\partial f(x,\alpha)}{\partial \alpha}\ \mathrm{d}x + f[h(\alpha),\alpha]\, \frac{\mathrm{d}h(\alpha)}{\mathrm{d}\alpha} - f[g(\alpha),\alpha]\, \frac{\mathrm{d}g(\alpha)}{\mathrm{d}\alpha}. \tag{81}$$

♣ Leibtnitz: n-ta derivacija umnoška

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} \left[f(x) \cdot g(x) \right] = \sum_{m=0}^{n} \binom{n}{m} \frac{\mathrm{d}^m f(x)}{\mathrm{d}x^m} \frac{\mathrm{d}^{n-m} g(x)}{\mathrm{d}x^{n-m}}. \tag{82}$$

♠

$$C^N + d^N = \prod_n \left(C + \mathrm{e}^{2\pi\imath n/N}\, d \right),$$

where

$$n = 1, 2, \cdots, N, \qquad N \ odd,$$
$$n = \frac{1}{2}, \frac{3}{2}, \cdots, \frac{N-1}{2}, \qquad N \ even.$$

♠

$$\ln(a+b) = \ln(b) + \ln\left[1 + \mathrm{e}^{\ln(a) - \ln(b)} \right].$$

♠

$$\cosh\left(\frac{K_1\,\sigma_1 + K_2\,\sigma_2}{2}\right) = A\;\mathrm{e}^{K\,\sigma_1\,\sigma_2}, \qquad \sigma_j = \pm 1,$$

$$A = \sqrt{\cosh\left(\frac{K_1+K_2}{2}\right)\;\cosh\left(\frac{K_1-K_2}{2}\right)},$$

$$K = \frac{1}{2}\;\ln\left[\frac{\cosh\left(\frac{K_1+K_2}{2}\right)}{\cosh\left(\frac{K_1-K_2}{2}\right)}\right].$$

♠ Kronecker δ function

$$\delta_{S_i,S_j} = \begin{cases} 1, & S_i = S_j \\ 0, & S_i \neq S_j \end{cases}, \qquad S_i, S_j = 0,1,2,\cdots,q-1,$$

$$\delta_{S_i,S_j} = \frac{1}{q}\sum_{k=0}^{q-1} S_i^k \cdot S_j^{-k}, \qquad S_i = \omega^n \in \mathbf{C}, \quad n = 0,1,2,\cdots,q-1$$

$$\omega = \mathrm{e}^{2\,\pi\,\imath/q}$$

$$\delta_{S_i,S_j} = \frac{1}{q}\sum_{k=0}^{q-1} \omega^{k\cdot(S_i-S_j)}, \qquad S_i = 0,1,2,\cdots,q-1 \in \mathbf{R}.$$

♠ Eulerova formula

$$\sum_{d=0}^{min\,\{D,s\}} \binom{D}{d}\binom{S-D}{s-d} = \binom{S}{s}. \tag{83}$$

... where we have used the following theorem involving binomial coefficients:

$$\sum_{k=0}^{n} \frac{(k+a)!}{k!a!} = \frac{(n+a+1)!}{n!(a+1)!}. \tag{84}$$

Primjedbe, komentari, sugestije, ...

Povratak na Predavanja.